Ⓘ Izračunati površinu lika koji je omeđen krivima $y \geq x^2+2x-3$, $y \leq -x$, $y \leq x+3$, $y \leq 0$.

Rj.: Skicirajmo date krive

$y = x^2+2x-3$ je parabola oblika $\cup$ sa tjemenom u $T(-1; -4)$

$y' = 2x+2$

$y' = 0$ akko $x = -1$

Izračunajmo presjek parabole $y = x^2+2x-3$ i pravih $y = -x$, $y = x+3$

$y = x^2+2x-3$
$y = -x$

$-x = x^2+2x-3$

$x^2+3x-3 = 0$

$D = 9+12$

$D = 21$

$x_{1,2} = \frac{-3 \pm \sqrt{21}}{2}$

$-3{,}8$
$0{,}8$

$\sqrt{21} \approx 4{,}58$

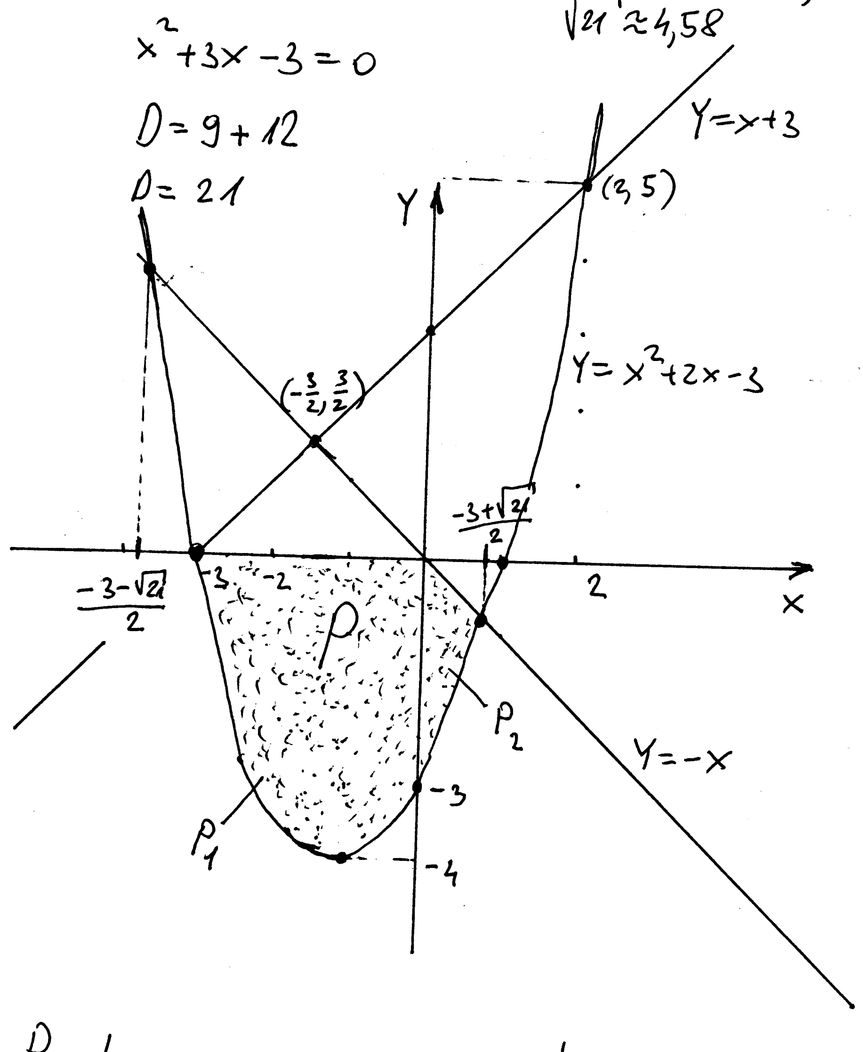
$y = x^2+2x-3$
$y = x+3$

$x^2+2x-3 = x+3$

$x^2+x-6 = 0$

$D = 1+24 = 25$

$x_1 = -3 \qquad x_2 = 2$



Nule parabole

$y = x^2+2x-3$

$x_1 = -3, \quad x_2 = 1$

$y = -x$
$y = x+3$

$2y = 3$

$y = \frac{3}{2} \Rightarrow x = -\frac{3}{2}$

Postoji više načina da se izračuna data površina. Najlakši način bi bio da podijelimo površinu na dva dijela

$$P = P_1 + P_2$$

$$P_1 = \int_{-3}^{0} |x^2+2x-3|\,dx = -\int_{-3}^{0} (x^2+2x-3)\,dx = \overset{\text{ZA VJEŽBU}}{\ldots} = 9$$

$$P_2 = \int_{0}^{\frac{-3+\sqrt{21}}{2}} \left((-x)-(x^2+2x-3)\right) dx = \int_{0}^{\frac{-3+\sqrt{21}}{2}} (3-3x-x^2)\,dx = \overset{\text{za vjež.}}{\ldots} = \frac{7\sqrt{21}}{4} - \frac{27}{4}$$

$$P = \frac{7\sqrt{21}}{4} + \frac{9}{4}$$ tražena površina lika

(#) Naći ekstreme f-je $z = x^3 + \frac{3}{2}y^2 - 3xy - 18y + 20$.

Rj.

$$\frac{\partial z}{\partial x} = 3x^2 - 3y$$

$$\frac{\partial z}{\partial y} = 3y - 3x - 18$$

$$\begin{aligned} 3x^2 - 3y &= 0 \quad /:3 \\ 3y - 3x - 18 &= 0 \quad /:3 \end{aligned}$$

$$\begin{aligned} x^2 - y &= 0 \quad \Rightarrow y = x^2 \\ y - x - 6 &= 0 \end{aligned}$$

$$x^2 - x - 6 = 0$$

$$(x+2)(x-3) = 0$$

$$x_1 = -2 \Rightarrow y_1 = 4$$

$$x_2 = 3 \Rightarrow y_2 = 9$$

Stacionarne tačke su $M(-2,4)$ i $N(3,9)$

Odredimo sad druge parcijalne izvode

$$\frac{\partial^2 z}{\partial x^2} = 6x$$

$$\frac{\partial^2 z}{\partial x \partial y} = -3$$

$$\frac{\partial^2 z}{\partial y^2} = 3$$

Za $M(-2,4)$ imamo $A = -12$, $B = -3$, $C = 3$

$$D = \begin{vmatrix} A & B \\ B & C \end{vmatrix} = AC - B^2 = -36 - 9 = -45 < 0 \Rightarrow$$ f-ja u tački M nema ekstrema

Za $N(3,9)$ imamo $A = 18$, $B = -3$, $C = 3$

$D = AC - B^2 = 45 > 0 \Rightarrow$ f-ja u tački N ima ekstrem

$A > 0 \Rightarrow$ f-ja ima minimum

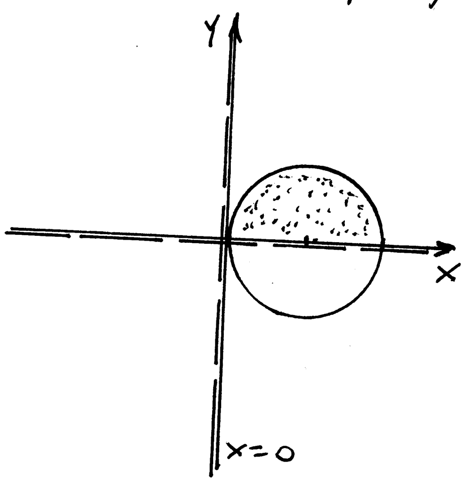
$$z_{MIN}(3,9) = 3^3 + \frac{3}{2} \cdot 9^2 - 3 \cdot 3 \cdot 9 - 18 \cdot 9 + 20 = -\frac{149}{2}$$

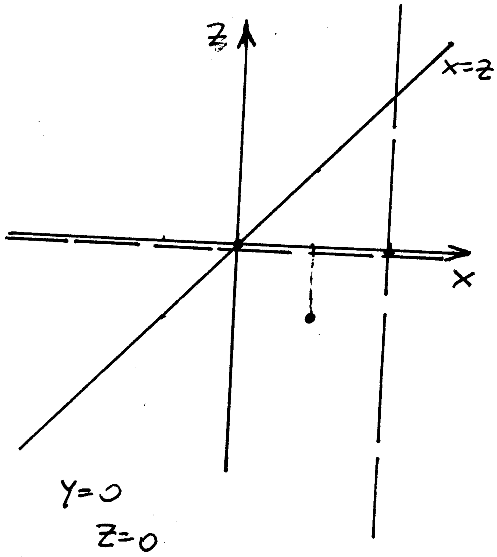
(#) Izračunati trostruki integral

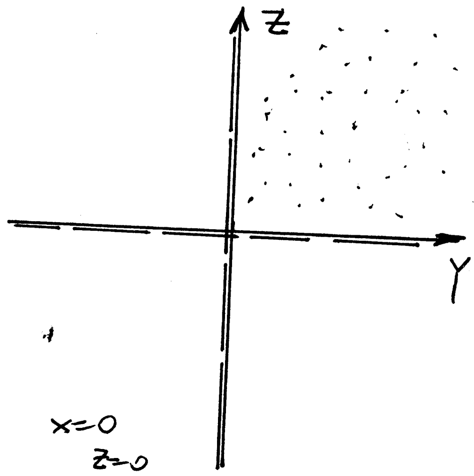
$$I=\iiint\limits_{\Omega} z\sqrt{x^2+y^2}\,dx\,dy\,dz$$

ako je oblast Ω omeđena ravnima $z\geq 0$, $z\leq x$, $y\geq 0$ i površinom $x^2+y^2\leq 2x$.

Rj.: Napravimo presjeke oblasti Ω sa xOy, xOz i yOz ravni.







$z=0$

$x=0$

$y=0$

$x^2+y^2=2x$

$x^2-2x+y^2=0$

$x^2-2\cdot x\cdot 1+1+y^2=1$

$(x-1)^2+y^2=1$

$y=0$

$z=0$

$z=x$

$y=0$

$x^2=2x$

$x(x-2)=0$

$x=0$ ili $x=2$

$x=0$

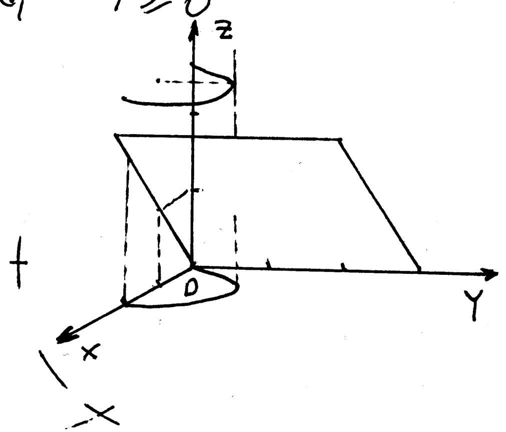
$z=0$

$z=0$

$y=0$

$y^2=0 \Rightarrow \cancel{y}=0$

Sa datih presjeka u stvari vidimo da je Ω unutrašnji dio cilindra koji se nalazi između ravni $z=0$ i $z=x$ i za koji još vrijedi $y\geq 0$.



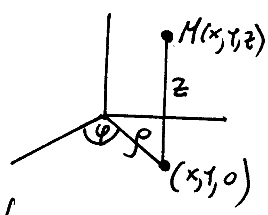
Uvedimo cilindrične koordinate

$x=\rho\cos\varphi$

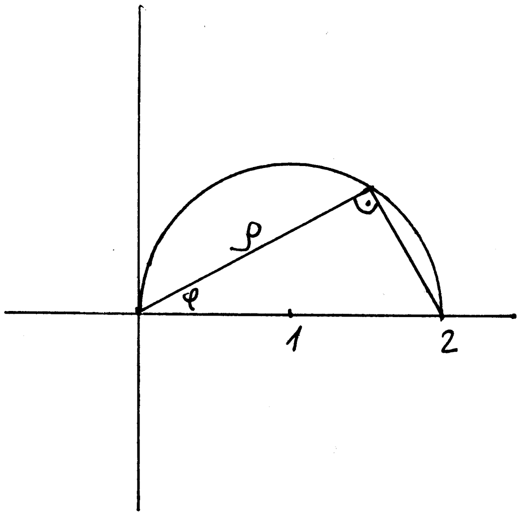
$y=\rho\sin\varphi$

$z=z$

$dx\,dy\,dz=\rho\,d\rho\,d\varphi\,dz$



opis tačke



$$\cos\varphi = \frac{\rho}{2}$$

$$\rho = 2\cos\varphi$$

$$D \xrightarrow{\text{transformiše}} D' : \begin{cases} 0 \le \rho \le 2\cos\varphi \\ 0 \le \varphi \le \frac{\pi}{2} \\ 0 \le z \le \rho\cos\varphi \end{cases}$$

$$x^2 + y^2 = \dots = \rho^2$$

Prema tome

$$I = \iiint\limits_{\Omega} z\sqrt{x^2+y^2}\,dx\,dy\,dz = \left| \begin{array}{l} \text{uvodimo} \\ \text{cilindrične} \\ \text{koordinate} \end{array} \right| =$$

$$= \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho^2\,d\rho \int_0^{\rho\cos\varphi} z\,dz = \int_0^{\pi/2} d\varphi \int_0^{2\cos\varphi} \rho^2 \underbrace{\frac{1}{2} z^2 \Big|_0^{\rho\cos\varphi}}_{\rho^2\cos^2\varphi} d\rho =$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\varphi \underbrace{\int_0^{2\cos\varphi} \rho^4\,d\rho}_{\frac{1}{5}\rho^5\big|_0^{2\cos\varphi}} = \frac{1}{2}\cdot\frac{1}{5}\int_0^{\pi/2} 2^5\cos^7\varphi\,d\varphi = \frac{16}{5}\int_0^{\pi/2}\cos^7\varphi\,d\varphi =$$

$= \ $ za vježbu ... $\ = \frac{256}{175}$ traženo rješenje

$$\cos^7\varphi = \cos^6\varphi\cdot\cos\varphi$$
$$= (\cos^2\varphi)^3\cdot\cos\varphi =$$
$$= (1-\sin^2\varphi)^3\cos\varphi$$

pa uvodimo smjenu

$$\sin\varphi = t$$
$$\cos\varphi\,d\varphi = dt$$

$\vdots$